CONCERNING THE SHAPE OF A GROWING DENDRITE

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Time-independent growth of a dendrite with a nonisothermic surface is considered. The Gibbs–Thomson correction, kinetic supercooling and the curvature correction under conditions of thermal balance are taken into account. The solid- and liquid-phase heat capacities need not be equal. We obtain lower estimates for the degree of difference of the dendrite shape from paraboloidal (parabolic), valid for any – not necessarily small – values of the parameters (supercooling, surface tension, etc.).

Crystallization progresses unstably when the heat produced by phase transition is removed through a supercooling melt and kinetic processes on the phase-separating surface are sufficiently fast. To determine the parameters of the resulting tree-like structure requires knowledge of the dimensions, shape and growth rate of each protuberance. Mathematical modeling of this problem usually considers steady growth of a single needle-shaped dendrite.

A simplified variation of this problem, in which the crystal surface is considered isothermal, has an infinite, continuous family of solutions for a given supercooling: paraboloids (parabolas) with one and the same product $v\rho_0$, dependent on the supercooling; this was shown by Ivantsov [11]. The resulting dependence agrees with experiment, but does not allow v and ρ_0 to be separately obtained. At the same time, it is known from experiments that for such growth are obtained a certain velocity v and radius of curvature ρ_0 of the point of the dendrite. There arises a problem of velocity selection, toward solution of which progress was made only in the middle 1980s. It turned out that even small surface tension and its small anisotropy are singular perturbations, and lead in a number of cases to selection of a finite number of solutions from the initial continuous Ivantsov family (for arbitrary supercooling, this was proved in [2]). Besides this, the growth velocity selection problem was solved analytically only in the limit of small deviations of the unknown dendrite surface from the Ivantsov paraboloid (parabola).

In the present work, we consider the question of the limits of applicability of such an assumption: we obtain lower estimates for the degree of difference of the dendrite shape from paraboloidal (parabolic), valid for any – not necessarily small – values of the parameters. Moreover, in addition to the Gibbs–Thomson correction, both the curvature correction in thermal balance and the kinetic processes on the phase-separating surface are taken into account. We also derive an integrodifferential equation describing the dendrite shape, of which a special case (with both liquid and gas phases having identical heat capacities, and with other simplifications) has been widely used in recent years, but, to the author's knowledge, without any proof.

Consider the crystal growth process with constant velocity v in the z-direction. Assume that the crystal surface S is smooth and divides space into two regions: U_1, filled by the crystal, and U_2, the melt. The temperature distribution $T = T_i$, $x \in U_i$, in a system of coordinates moving with speed v is described by the time-independent heat conduction equation

$$D\nabla^2 T + v\frac{\partial T}{\partial z} = 0. \quad (1)$$

The condition of thermal balance on S has the form [4]

$$D\left[c_1\frac{\partial T_1}{\partial n} - c_2\frac{\partial T_2}{\partial n}\right] = v_n\left[L - (\varepsilon + T_{\mathrm{melt}}L^{-1}(c_2 - c_1)\gamma)\,\rho^{-1}(\mathbf{x})\right]. \quad (2)$$

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The temperature at the crystallization front is given by [3]

$$T = T_{\text{melt}} - K^{-1}v_n - T_{\text{melt}} L^{-1}\gamma\rho^{-1}(\mathrm{x}). \tag{3}$$

Far from the front, the melt is supercooled: $\lim\limits_{z\to+\infty} T = T_{\text{melt}} < T_\infty$. In addition, there is a constraint on the behavior of T at infinity: $\max\{|T - T_\infty| + |\nabla T|\} < \infty$, and in the three-dimensional case, within any convex-down paraboloid with vertical axis of symmetry, it must hold that $|T - T_\infty| + |\nabla T|\} \to 0$ for $z \to +\infty$ uniformly in x and y.

Introducing the length scale $l = 2D/v$, for the nondimensional temperature $t = (T - T_\infty)c_2L^{-1}$, we obtain the system of equations

$$\nabla^2 t + 2\frac{\partial t}{\partial z} = 0, \tag{4}$$

$$\beta\frac{\partial t_1}{\partial n} - \frac{\partial t_2}{\partial n}\bigg|_{S'} = 2\left(1 - \frac{d_1}{l\rho'}\right)n_z, \tag{5}$$

$$t|_{S'} = \Delta - \alpha v_n - \frac{d_0}{l\rho'}, \tag{6}$$

$$\lim_{z\to+\infty} t = 0, \quad |t| + |\nabla t| \leqslant M. \tag{7}$$

Here $\beta = c_1/c_2$; $\Delta = (T_{\text{melt}} - T_\infty)c_2L^{-1}$ is the nondimensional supercooling; $\alpha = c_2/KL$; $d_0 = c_2T_{\text{melt}}\gamma/L^2$; $d_1 = \varepsilon/L + (1 - \beta)d_0$; $S' = S/l$ is the nondimensional surface, whose radius of curvature is $\rho' = v\rho/2D$.

Now let T satisfy the thermal conduction equation (1) and the boundary conditions at infinity. We fix an arbitrary point $\mathrm{x}_0 \in S$ and consider the fundamental solution of Eq. (4):

$$g(\mathrm{x}) = \exp(z - z_0 - R)/2\pi R,$$

where $R = |\mathrm{x} - \mathrm{x}_0|$. In Green's formula for any finite region $U \subset U_i' = U_i/l$, whose closure does not contain x_0:

$$\int\limits_{\partial U}\left(g\frac{\partial t}{\partial n} - t\frac{\partial g}{\partial n} + 2gtn_z\right)ds = 0. \tag{8}$$

We take $U = U_i' \cap [B(\mathrm{x}_0, r) - B(\mathrm{x}_0, \varepsilon)]$, where $B(\mathrm{x}, a)$ is a sphere of radius a with center at x. We evaluate the integral over $U_i' \cap \partial B(\mathrm{x}_0, r)$ for large r,

$$\left|\int\limits_{\partial U}\left(g\frac{\partial t}{\partial n} - t\frac{\partial g}{\partial n} + 2gtn_z\right)ds\right| \leqslant 5\int\limits_{\partial U}(|t| + |\nabla t|)g\,ds \leqslant 5\int\limits_{\partial U}(|t| + |\nabla t|)\exp(z - z_0 - r)\,dz.$$

The integral over the intersection of ∂U with the paraboloid $r - (z - z_0) \le C$ is less than $C \max\{|t| + |\nabla t|, r - (z - z_0) \le C\}$ and tends to zero for fixed C and $r \to \infty$. However, the integral over remainder of ∂U is less than $M\exp(-C)$ and becomes arbitrarily small for sufficiently large C. Therefore, the integral over $U_i' \cap \partial B(\mathrm{x}_0, r)$ tends to zero as $r \to \infty$. Substituting U into (8) and taking limit as $r \to \infty$, $\varepsilon \to 0$,

$$\int\limits_{S'}\left(g\frac{\partial t_i}{\partial n_i} - t\frac{\partial g}{\partial n_i} + 2gtn_{iz}\right)ds = t(\mathrm{x}_0), \quad i = 1, 2,$$

where n_i is the outward normal to U_i' ($n_1 = n$, $n_2 = -n$). Multiplying the first of the two obtained equations by β and adding the result to the second,

$$(\beta + 1)t(\mathrm{x}_0) = \int\limits_{S'}\left[\left(\beta\frac{\partial t_1}{\partial n} - \frac{\partial t_2}{\partial n}\right)g(\mathrm{x}, \mathrm{x}_0) + (\beta - 1)t(\mathrm{x})h(\mathrm{x}, \mathrm{x}_0)\right]ds. \tag{9}$$

Here $h = 2gn_z - \partial g/\partial n = [n_z + (1 + R^{-1})n_R]g$. Now substituting the thermal balance equation (5) into (9),

$$\frac{\beta + 1}{2}t(\mathrm{x}_0) = \int\limits_{S'}\left(1 - \frac{d_1}{l\rho'}\right)g(\mathrm{x}, \mathrm{x}_0)\,dxdy + \frac{\beta - 1}{2}\int\limits_{S'}t(\mathrm{x})h(\mathrm{x}, \mathrm{x}_0)\,ds. \tag{10}$$

We have obtained the necessary condition for growth of S with constant speed v, valid for any boundary condition on the assumed value of t on S' (corresponding to T on S). In particular, when the t, obtained from the boundary condition (7), is substituted into (10), the result is an equation describing all solutions of the system (4)-(7).

If we take S' in (10), symmetric relative to displacements in the y-direction, we obtain

$$\frac{\beta+1}{2}t(\mathbf{x}_0)=\int_{S_2'}\left(1-\frac{d_1}{l\rho'}\right)g_2(\mathbf{x},\ \mathbf{x}_0)\,dx+\frac{\beta-1}{2}\int_{S_2'}t(\mathbf{x})\,h_2(\mathbf{x},\ \mathbf{x}_0)\,ds. \tag{11}$$

Here S_2' is the projection of S' onto the x–z plane, $r=\sqrt{(x-x_0)^2+(z-z_0)^2}$,

$$g_2(\mathbf{x},\ \mathbf{x}_0)=\int_{-\infty}^{+\infty}g(\mathbf{x},\ \mathbf{x}_0)\,dy=\pi^{-1}\exp(z-z_0)K_0(r),$$

$$h_2(\mathbf{x},\ \mathbf{x}_0)=2g_2n_z-\partial g_2/\partial n=\pi^{-1}\exp(z-z_0)[n_zK_0(r)+n_rK_1(r)].$$

Equation (11) describes the time-independent growth of a two-dimensional dendrite.

We now make use of Eq. (10) to determine the difference in shape of a needle-shaped dendrite from paraboloidal. Briefly, the idea of the calculation is as follows: The integrals on the right-hand side of (1) may be estimated using analogous integrals over two paraboloids, the larger of which contains S' while the smaller is contained within S'. The smalle the difference of S' from these two paraboloids, the closer the right-hand side of (10), as a function of S', is to a constant. But we know the degree of difference of the left-hanc side from constant; it is given by the term $\alpha v_n + d_0/l\rho'$ in (6). In this way, we may obtain a lower estimate for the difference between the radii of curvature of the exterior and the interior paraboloids.

Thus, we place the coordinate origin at the vertex of S': $z = z(r,\ \varphi)$ and define

$$\rho_1=\inf_S\frac{\partial(r^2)}{\partial(-2z)},\quad \rho_2=\sup_S\frac{\partial(r^2)}{\partial(-2z)},\quad \rho_i'=\frac{\rho_i}{l},$$

$$\mu_1=\inf_S\frac{-2z}{r^2}\left[\frac{\partial(r^2)}{\partial(-2z)}\right]^2,\quad \mu_2=\sup_S\frac{-2z}{r^2}\left[\frac{\partial(r^2)}{\partial(-2z)}\right]^2,\quad \mu_i'=\frac{\mu_i}{l}. \tag{12}$$

Further, consider the case of finite, positive ρ_1 and ρ_2 (and μ_1, μ_2, such that $\rho_1^2/\rho_2 \le \mu_1 \le \rho_0 \le \rho_2 \le \mu_2 \le \rho_2^2/\rho_1$). Thus, the dendrite surface S is confined between two paraboloids of revolution $\Pi(\rho_1)$ and $\Pi(\rho_2)$ [hence between $\Pi(\mu_1)$ and $\Pi(\mu_2)$], where $\Pi(\rho)$ is specified by the equation $z = -r^2/2\rho$.

Let ρ_0 denote the radius of curvature of S at the vertex; ρ_{min}, the minimum radius of curvature of S; and $t_0 = \Delta - \alpha v - d_0/\rho_0$ the nondimensional temperature at the vertex of S.

Because any paraboloid of revolution is a solution of the isothermal Stefan problem [1]

$$\int_{\Pi(\rho)}g(\mathbf{x},\ \mathbf{x}_0)\,rdrd\varphi\equiv I(\rho),$$

i.e., I depends only on ρ and not on x_0. Substituting $x_0=0$ and integrating, we obtain $I(\rho) = \rho f(\rho)$, $f(\rho) = \exp(\rho)E_1(\rho)$.

We now estimate the first integral in the right-hand side of (11). For $z_0 = 0$ $g(\mathbf{x},0)_{S'}\geqslant g(\mathbf{x},0)_{\Pi(\rho_1')}$, if we compare the values of the function at points with the same projections onto the horizontal plane. Therefore,

$$\int_{S'}\left(1-\frac{d_1}{l\rho'}\right)g(\mathbf{x},\ 0)\,rdrd\varphi\geqslant\left(1-\frac{d_1}{\rho_{\min}}\right)\int_{\Pi(\rho_1')}g(\mathbf{x},\ 0)\,rdrd\varphi=\left(1-\frac{d_1}{\rho_{\min}}\right)\rho_1'f(\rho_1'). \tag{13}$$

On the other hand,

$$\left.\frac{\partial r}{\partial(-z)}\right|_{S',z=z_0}=\frac{1}{r}\frac{\partial(r^2)}{\partial(-2z)}\geqslant\sqrt{\frac{\mu_1'}{-2z_0}}=\left.\frac{\partial r}{\partial(-z)}\right|_{\Pi(\mu_1'),z=z_0},$$

and consequently, for any z, $z_0 \le 0$ $|r-r_0|_{S'}\ge|r-r_0|_{\Pi(\mu_1')}$, $R|_{S'}\geqslant R|_{\Pi(\mu_1')}$ and $g(\mathbf{x},\mathbf{x}_0)_{S'}\leqslant g(\mathbf{x},\mathbf{x}_0)_{\Pi(\mu_1')}$, if the values are compared at points $\mathbf{x}|_{S'}$ and $\mathbf{x}|_{\Pi(\mu_1')}$, $\mathbf{x}_0|_{S'}$ a r $\mathbf{x}_0|_{\Pi(\mu_1')}$ with the same corresponding projections onto the z-axis. Therefore,

$$\int_{S'}\left(1-\frac{d_1}{l\rho'}\right)g(\mathbf{x},\ \mathbf{x}_0)\,r\,dr\,d\varphi\leqslant\rho_2'\int_{S'}g(\mathbf{x},\ \mathbf{x}_0)\,dz\,d\varphi\leqslant\rho_2'\int_{\Pi(\mu_1')}g(\mathbf{x},\ \mathbf{x}_0)\,dz\,d\varphi=\rho_2'f(\mu_1'). \tag{14}$$

In particular, (14) is valid for $z_0 \to -\infty$. Substituting the inequality (14) with $z_0 \to -\infty$ into (13) and (10), we find

$$\frac{\rho_2'f(\mu_1')}{\rho_1'f(\rho_1')}\geqslant\left(1-\frac{d_1}{\rho_{\min}}\right)\left(\frac{\beta+1}{2}\Delta-J_{\max}\right)\Big/\left(\frac{\beta+1}{2}t_0-J_{\min}\right),$$

where J_{max} and J_{min} are the extremal values of the second term in (10). To evaluate these, we make use of

$$\int_{S'}h(\mathbf{x},\ \mathbf{x}_0)\,ds\equiv 1,$$

which results from (8) with the substitution t_1 = const [so that t_1 satisfies Eq. (4)], as well as $h(x, x_0) \geq 0$. Consequently,

$$\Delta-\alpha v-d_0/\rho_{\min}\leqslant t_{\min}\leqslant\int_{S'}t(\mathbf{x})\,h(\mathbf{x},\ \mathbf{x}_0)\,ds\leqslant t_{\max}=\Delta.$$

Because J_{max} and J_{min} differ from the integral of the latter estimate by the factor $(\beta - 1)/2$,

$$J_{\max}-J_{\min}\leqslant\left|\frac{\beta-1}{2}(\Delta-t_{\min})\right|\leqslant\left|\frac{\beta-1}{2}\right|(\alpha v+d_0/\rho_{\min}),$$

$$\frac{\beta+1}{2}\Delta-J_{\max}\geqslant\frac{\beta+1}{2}\Delta-\frac{\beta-1}{2}\Delta=\Delta.$$

Using these inequalities, we obtain

$$\frac{\rho_2'f(\mu_1')-\rho_1'f(\rho_1')}{\rho_2'f(\mu_1')}\geqslant\left[\left(1-\frac{d_1}{\rho_{\min}}\right)\left(\frac{\beta+1}{2}\Delta-J_{\max}\right)-\right.$$

$$\left.-\left(\frac{\beta+1}{2}t_0-J_{\min}\right)\right]\Big/\left[\frac{\beta+1}{2}\Delta-J_{\max}\right]\geqslant\frac{av+d/\rho_0-c/\rho_{\min}}{\Delta},$$

where $a = \alpha b$; $b = \min\{\beta, 1\}$; $d = d_0\ (\beta + 1)/2$; $c = d_0\ |1 - \beta|/2 + d_1\Delta$.

We now find the majorant of the left-hand side of the latter inequality in the form of a product $\Delta\rho$ of several factors:

$$f(\mu_1')-f(\rho_1')\leqslant(\rho_1'-\mu_1')\max_{\mu_1'\leqslant x\leqslant\rho_1'}[-df/dx]=(\rho_1'-\mu_1')\left(\frac{1}{\mu_1'}-f(\mu_1')\right)\leqslant\frac{\Delta\rho}{\rho_1}(1-\mu_1'f(\mu_1'));$$

$$\frac{\rho_2'f(\mu_1')-\rho_1'f(\rho_1')}{\rho_2'f(\mu_1')}=\frac{\Delta\rho}{\rho_2}+\frac{\rho_1}{\rho_2}\frac{f(\mu_1')-f(\rho_1')}{f(\mu_1')}\leqslant\frac{\Delta\rho}{\rho_2}\left[1+\frac{1}{f(\mu_1')}-\mu_1'\right].$$

We prove that the function

$$\eta(x)=1/f(x)-x=\exp(-x)/E_1(x)-x,\ x>0,\ 0<\eta(x)<1, \tag{15}$$

monotonically increases. Define

$$u(x)=\exp(-x)\,xf^2\eta'=\exp(-x)(1-xf-xf^2).$$

Its derivative $u' = \exp(-x)(f - f^2 - xf^2) < 0$, because

$$f(x)=\frac{1}{x}\int_0^\infty\frac{\exp(-t)}{1+t/x}dt>\frac{1}{x}\int_0^\infty\frac{\exp(-t)}{\exp(t/x)}dt=\frac{1}{1+x}.$$

Consider the asymptotic behavior of $u(x)$ as $x \to \infty$:

$$f(x)=x^{-1}-x^{-2}+2x^{-3}-6x^{-4}+O(x^{-5}),$$

$$u(x)=\exp(-x)(x^{-3}+O(x^{-4}))>0$$

for large x. Consequently, $u(x) > 0$ and $\eta'(x) > 0$, from which it follows that $\eta(x)$ is monotonic.

Using this, we obtain the following estimate:

$$\frac{\Delta\rho}{\rho_2} \geqslant \frac{av + d/\rho_0 - c/\rho_{\min}}{(1+\eta(p))\Delta} \geqslant \frac{av + d/\rho_0 - c/\rho_{\min}}{2\Delta}. \tag{16}$$

Here $p = v\rho_0/2D$ is the Peclet number.

The second inequality of this chain is applicable for large supercooling, when $p \to \infty$ and $\eta(p) \to 1$. For small supercooling ($p \to 0$), $\eta(p) \sim \ln^{-1}(p^{-1})$.

The estimate (16) may be written differently:

$$\Delta\rho \geqslant \frac{(2aD)p + (d - c\rho_2/\rho_{\min})}{(1+\eta(p))\Delta}. \tag{17}$$

For small α, d_0, d_1 $\Delta \approx I(p)$ and $(1 + \eta(p))\Delta \approx \Delta + p(1 - \Delta)$.

The vast majority of works analyzing time-independent growth of single dendrites consider the symmetric model ($\beta = 1$) with null coefficient d_1. In that case the estimates (16) and (17) simplify:

$$\frac{\Delta\rho}{\rho_2} \geqslant \frac{\alpha v + d_0/\rho_0}{(1+\eta)\Delta}, \tag{18}$$

$$\Delta\rho \geqslant \frac{(2\alpha D)p + d_0}{(1+\eta)\Delta}. \tag{19}$$

For small values of α, d_0, and d_1, S is nearly a parabola and $\rho_{\min} \approx \rho_0$. In the general case, if $\beta \neq 1$, while $\rho_{\min}$ is considerably less than ρ_0, estimates (16) and (17) may lose their meaning, because the negative addition on the right-hand side becomes too large. We may then advance in the following way: substitute inequality (13) in an analogous way, taking x_0 as a point on the front possessing the minimum temperature $t_{\min}$:

$$\int_{S'} \left(1 - \frac{d_1}{l\rho'}\right) g(x, x_0)\, r dr d\varphi \geqslant \left(1 - \frac{d_1}{\rho_{\min}}\right) \rho_1' f(\mu_2'). \tag{20}$$

Substituting (14) and (20) into (10), we obtain

$$\frac{\rho_2' f(\mu_1') - \rho_1' f(\mu_2')}{\rho_2' f(\mu_1')} \geqslant b\frac{\Delta - t_{\min}}{\Delta} - \frac{d_1}{\rho_{\min}}.$$

We estimate the left-hand side of the last inequality:

$$f(\mu_1') - f(\mu_2') \leqslant (\rho_2' - \rho_1') \max_{\rho_1' \leqslant x \leqslant \rho_2'} \frac{d}{dx}\left[f(\rho_1'^2/x) - f(x^2/\rho_1')\right] \leqslant \frac{\Delta\rho}{\rho_1}\left(3 - 3\mu_1' f(\mu_1')\right);$$

$$\frac{\rho_2' f(\mu_1') - \rho_1' f(\mu_2')}{\rho_2' f(\mu_1')} = \frac{\Delta\rho}{\rho_2} + \frac{\rho_1}{\rho_2}\frac{f(\mu_1') - f(\mu_2')}{f(\mu_1')} \leqslant \frac{\Delta\rho}{\rho_2}(1 + 3\eta(\mu_1')).$$

Thus,

$$(1 + 3\eta(p))\frac{\Delta\rho}{\rho_2} \geqslant b\frac{\Delta - t_{\min}}{\Delta} - \frac{d_1}{\rho_{\min}} \geqslant \frac{bd_0 - d_1\Delta}{\rho_{\min}\Delta}. \tag{21}$$

For $d_1 = 0$ we obtain the estimate

$$\frac{\Delta\rho}{\rho_2} \geqslant b\frac{\alpha v + d_0/\rho_{\min}}{(1+3\eta)\Delta}. \tag{22}$$

We have thus obtained lower estimates for the relative [Eqs. (16), (18), (21) and (22)] and absolute [Eqs. (17) and (19)] differences of the dendrite shape from paraboloidal.

All the preceding calculations were performed with the assumption that the free and interior surface energies γ and ε and the linear kinetic coefficient K are isotropic. However, it is obvious that the derivation of the principal integrodifferential equations (10) and (11) and of the estimates, (16)-(22), is also valid for anisotropic γ, ε and K. Moreover, in (10) and (11), α, d_0 and d_1 appear as specified functions on S'; in estimates (16)-(22), everywhere that d_0 and d_1 are coefficients for ρ_0^{-1}, they take their values at the peak of the dendrite; instead of α, wherever it takes its maximum value on S'.

To analyze the two-dimensional problem of estimation of integrals (14) and (20), we make the following modification:

$$\int_{S'} g(\mathbf{x},\ \mathbf{x}_0)\,dx = \int_{x\leqslant 0} + \int_{x\geqslant 0} g(\mathbf{x},\ \mathbf{x}_0)_{S'}\,\frac{dx}{d(-z)}\,d(-z)\leqslant$$

$$\leqslant \int_{x\leqslant 0} + \int_{x\geqslant 0} g(\mathbf{x},\ \mathbf{x}_0)_{S'}\sqrt{\frac{\mu_2'}{-2z}}\,d(-z)\leqslant \sqrt{\frac{\mu_2'}{\mu_1'}}\int_{\Pi(\mu_1')} g(\mathbf{x},\ \mathbf{x}_0)\,dx = \sqrt{\mu_1'\mu_2'}\,f(\mu_1');$$

$$\int_{S'} g(x,\ x_0)\,dx \geqslant \sqrt{\mu_1'\mu_2'}\,f(\mu_2').$$

Estimates (16)-(22) remain valid also in the two-dimensional case with the replacement of $\eta(p)$ by

$$\eta_2(p) = \sqrt{p/\pi}\exp(-p)/\mathrm{Erfc}(\sqrt{p}) - p,\quad 0<\eta_2(p)<1/2,$$

while in estimates (21) and (22), $1 + 3\eta$ must be replaced by $3/2 + 3\eta_2$.

The physical meaning of the derived results is the following: to the first approximation, the lower estimate for the relative difference $\Delta\rho/\rho_2$ of the dendrite shape from paraboloidal for a given supercooling depends linearly on v and ρ_0^{-1}. In the symmetric model, for example, if by some other means the degree of paraboloidalness $\Delta\rho/\rho_2 \le \nu$ of the dendrite becomes known, then one may say that the growth rate of such a dendrite is not greater than v_*, $v_* = \nu\Delta/(\alpha + d_0/2Dp)$, while the radius of curvature of the point is not less than ρ_*, $\rho_*^{-1} = \nu\Delta/(2\alpha Dp + d_0)$. From the unsymmetric model are obtained analogous estimates for v and ρ_{min}. However, in order to estimate the absolute difference $\Delta\rho$, it is sufficient to know only the supercooling Δ, because for small α, d_0 and d_1 it is not difficult to obtain a lower estimate for the dependence of p on Δ.

The obtained estimates may be useful for determining in a given case whether the dendrite may be considered a paraboloid (parabola) to a sufficient degree of accuracy, which is as important in practical work as in theory.

This is illustrated by the example of growth of a dendrite of tin. In tin, T_{melt} = 505 K, L = 4.26×10^8 J/m³, $c_2 = 1.83 \times 10^6$ J/(m³·K), $c_1 > c_2$, $d_1 = 0$, $\gamma = 0.109$ J/m² [3], and K = 0.116 m/(K·sec) [5]. For the supercooling $T_{melt} - T_\infty = 10.5$ K, the dendrite grows with speed v = 0.25 m/sec [3], and from inequality (18) we obtain $\Delta\rho/\rho_2 \ge 21\%$, and the contribution in the estimate from the term with kinetic coefficient K in this case is 25 times larger than that of the free energy γ. Nevertheless, in the majority of works investigating the correction to the parabolic dendrite shape and the resulting discrete velocity distribution, the influence of surface tension is limited, disregarding supercooling at the crystallization front. From the above, it follows that such simplification is evidently not correct enough. For the value obtained in [6], v = 0.8 m/sec for $T_{melt} - T_\infty = 11$ K, we obtain $\Delta\rho/\rho_2 \ge$ 65%.

Let us now see what the obtained estimates give for the theory of microscopic solutions. In this theory, it is assumed that $\beta = 1$ and $\alpha = d_1 = 0$, and from the linearized differential equation for the form of the surface for any Δ, $0 < \Delta < 1$, [2] concludes that the so-called LM – K parameter $\sigma = d_0/p\rho_0$ is a function of the anisotropy and does not depend on the supercooling or on the other parameters of the problem. In [7] are obtained experimental results for σ for various Δ: from 0.02 to 0.03. Substituting $d_0/\rho_0 = \sigma p$ into (18), we obtain $\Delta\rho/\rho_2 \ge \sigma p/2\Delta$. Consequently, the theory of microscopic solutions is inapplicable for large Δ (p must be small in comparison with σ^{-1}).

In [8], for linearization of the symmetric model in the limit of small p, the influence of surface tension and kinetic supercooling and its anisotropy on the selection of growth rate of a two-dimensional dendrite is analyzed. In particular, in the limit of isotropic kinetics, when the anisotropy coefficient of α is small in comparison with ε_d, the anisotropy coefficient of d_0, an asymptote is obtained for the growth rate $\alpha v \sim (\varepsilon_d k)^{5/6} p^{1/6}$ ($k = d_0/2D\alpha$ is a nondimensional parameter). Substituting this expression into (18), we obtain for $p \to 0$, $\Delta\rho/\rho_2 \ge (\varepsilon_d k)^{5/6}\,(p^{-1/3} + kp^{-4/3}) \to \infty$, and the degree of nonparabolicity of the dendrite grows without bound.

NOTATION

T) temperature; v) growth rate; z) direction of growth; S) crystal surface; n) normal to S; $\rho(x)$) radius of curvature of S; x) radius vector (x, y, z); (r, φ)) polar coordinates in the x–y plane; D) thermal conductivity; c_i, i = 1, 2) heat capacity; L) latent heat of melting; ε) surface internal energy; γ) surface free energy; T_{melt}) melting temperature of a planar crystal; K) linear kinetic coefficient; $l = 2D/v$) length scale; $t = (T - T_\infty)c_2L^{-1}$) nondimensional temperature; $p = v\rho_0/2D$) Peclet number. The subscripts i = 1, 2 denote the solid and liquid phases respectively; primes denote nondimensional quantities.

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DETERMINING THE THERMAL CONDUCTIVITY OF CERAMIC MATERIALS BY SOLVING THE INVERSE HEAT-CONDUCTION PROBLEM

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A method is propounded for determining the temperature dependence of the thermal conductivity of ceramic materials. It is based on a solution of the inverse heat-conduction problem. An installation is described for carrying out the thermophysical experiment. The temperature dependence of thermal conductivity of a ceramic material has been obtained.

The implementation of effective high-temeprature processes and equipment depends to a large extent on investigations of materials, and a considerable part of this consists of exploring the thermophysical characteristics of the materials. Such investigations, as a rule, reduce to solving a number of complex technical and mathematical problems. This applies to the experimental-computational determination of thermal conductivity, although here it is usually not necessary that a temperature field varying according to a specified program be maintained, it being sufficient to ensure a stable steady-state heat transfer. Nonetheless, growing demands for precision in the determination of thermal conductivity over a wide temperature range make it necessary to perfect both the techniques of the thermal experiment and the processing of the results. Very promising in this connection is the use of methods of solving the inverse heat-conduction problem (IHCP) [1, 2], allowing a widening of the range of variation of thermal loading of the specimen, which is suitable for the indirect measurement of thermal conductivity. This makes it possible to lower the demand for precise experimental data, and to raise the quality of identifying the thermal conductivity, by simultaneous processing of information received from a large number of points at which temperature is monitored, or under the conditions of an increased number of variants of thermal loading. Advantages of the IHCP methods consist of being able to take into account properly and without special difficulties, the dependence of thermophysical properties on

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